EXERCISE – III — HINTS & SOLUTIONS

Sol.1 **(i)** $I = \int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$

$= \int_{-\infty}^{\infty} \frac{dx}{1+(x+1)^2} = \tan^{-1}(x+1)\Big|_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$

(ii) $I = \int_{\sqrt{2}}^{\infty} \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x \Big|_{\sqrt{2}}^{\infty} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

(iii) $I = \int_0^4 \frac{x^2+1-1}{x+1} dx$

$= \int_0^{-1}(x-1)\,dx + \int_0^4 \frac{dx}{1+x} = \frac{x^2}{2} - x\Big|_0^4 + \ell n(1+x)\Big|_0^4$

$= \frac{16}{2} - 4 + \ell n 5 = 4 + \ell n\, 5$

Sol.2 $f(x) = \ell n\left(\frac{1-\sin x}{1+\sin x}\right)$

$f(-x) = \ell n\left(\frac{1+\sin x}{1-\sin x}\right) = -\ell n\left(\frac{1-\sin x}{1+\sin x}\right) = -f(x)$

Odd function

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$= -\int_b^a \ell n\left(\frac{1-\sin x}{1+\sin x}\right)dx = \int_b^a \ell n\left(\frac{1+\sin x}{1-\sin x}\right)dx$

Sol.3 **(i)** $I = \int_b^2 [x^2]\,dx$

Put $x^2 = t$

$x\,dx = \frac{dt}{2}$

$dx = \frac{dt}{2\sqrt{t}} = \frac{1}{2}\int_0^4 \frac{[t]}{\sqrt{t}}\,dt$

$= \frac{1}{2}\left[\int_0^1 \frac{0}{\sqrt{t}}dt + \int_1^2 \frac{dt}{\sqrt{t}} + 2\int_2^3 \frac{dt}{\sqrt{t}} + 3\int_3^4 \frac{dt}{\sqrt{t}}\right]$

$= \frac{1}{2}[(2\,t^{1/2})^2{}_1 + 4\,(\sqrt{t}\,)_2{}^3 + 6(\sqrt{t}\,)_3{}^4]$

$= \frac{1}{2}\,[2\sqrt{2} - 2 + 4\sqrt{3} - 4\sqrt{2} + 6 \times 2 - 6\sqrt{3}\,]$

$= \frac{1}{2}\,[10 - 2\sqrt{3} - 2\sqrt{2}\,] \quad = 5 - \sqrt{3} - \sqrt{2}$

(ii) $I = \int_{-1}^{1} [\cos^{-1} x]\,dx$

$\cos^{-1} x = t$
$x = \cos t$
$dx = -\sin t\, dt$

$= -\int_{\pi}^{0} [t]\sin t\,dt = \int_0^{\pi} [t]\sin t\,dt$

$= \int_0^1 0.\sin t dt + \int_1^2 1.\sin t dt + \int_2^3 2.\sin t dt + \int_3^{\pi} 3.\sin t dt$

$= -[\cos t|_1^2 + 2\cos t|_2^3 + 3\,\cos t|_3^{\pi}]$

$= -[\cos 2 - \cos 1 + 2\cos 3 - 2\cos 2 - 3 - 3\cos 3]$

$= -[-\cos 2 - \cos 1 - \cos 3 - 3]$

$= [3 + \cos 1 + \cos 1 + \cos 3]$

Sol.4 **(i)** $I = \int_{-1}^{1} e^{|x|}dx$

$= \int_{-1}^{0} e^{-x}dx + \int_0^1 e^x dx = -e^{-x}\Big|_{-1}^{0} + e^x\Big|_0^1$

$= -[1 - e] + [3 - 1] \quad = 2e - 2$

(ii) $I = \int_{-\pi/4}^{\pi/4} |\sin x|\,dx$

$I = \int_{-\pi/4}^{0} -\sin x\,dx + \int_0^{\pi/4} \sin x\,dx$

$= \cos x\,|_{-\pi/4}^{0} - \cos x\,|_0^{\pi/4}$

$$= \left(1-\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{2}}-1\right) = 2 - \sqrt{2}$$

(iii) $$I = \int_{-3}^{3} |x+2|\,dx$$

$$= \int_{-5}^{-2} |x+2|\,dx + \int_{-2}^{5} |x+2|\,dx$$

$$= -\int_{-5}^{-2} (x+2)dx + \left.\frac{x^2}{2}+2x\right|_{-2}^{5}$$

$$= -\left[2-4-\frac{25}{2}+10\right] + \left[\frac{25}{2}+10-2+4\right]$$

$$= -8 + \frac{25}{2} + \frac{25}{2} + 12 \quad = 4 + 25 \quad = 29$$

(iv) $$I = \int_{-\pi/4}^{\pi/4} \frac{x+\frac{\pi}{4}}{2-\cos 2x}\,dx$$

$$= \underset{\text{Odd Function}}{\underset{\downarrow}{\int_{-\pi/4}^{\pi/4} \frac{x\,dx}{2-\cos 2x}}} + \frac{\pi}{4} \underset{\text{Even function}}{\underset{\downarrow}{\int_{-\pi/4}^{\pi/4} \frac{dx}{2-\cos 2x}}}$$

$$= 0 + \frac{\pi}{2} \int_{0}^{\pi/4} \frac{dx}{2-\cos 2x}$$

$$= \frac{\pi}{2} \int_{0}^{\pi/4} \frac{1+\tan^2 x}{2(1+\tan^2 x)-(1-\tan^2 x)}$$

$$= \frac{\pi}{2} \int_{0}^{\pi/4} \frac{\sec^2 x\,dx}{1+3\tan^2 x}$$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$= \frac{\pi}{2} \int_{0}^{1} \frac{dt}{1+3t^2} \quad = \frac{\pi}{6} \int_{0}^{1} \frac{dt}{1+3t^2}$$

$$= \frac{\pi}{6}\sqrt{3}\ \tan^{-1} + \sqrt{3}\,\Big|_0^1 = \frac{6\pi}{6}\ [\tan^{-1}\sqrt{3} - 0]$$

$$= \sqrt{3}\,\frac{\pi}{6} \times \frac{\pi}{3} = \frac{\pi^2\sqrt{3}}{18} = \frac{\pi^2}{6\sqrt{3}}$$

Sol.5 **(i)** $$I = \int_{0}^{1} \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$$

Put $x = \tan\theta$

$dx = \sec^2\theta\,d\theta$

$$= \int_{0}^{\tan^{-1}1} \sin^{-1}(\sin 2\theta)\sec^2\theta\,d\theta \qquad 0 < \theta < \frac{\pi}{4}$$

$$= \int_{0}^{\pi/4} 2\theta\,.\,\sec^2\theta\,d\theta \qquad 0 < 2\theta < \frac{\pi}{4}$$

$$= 2\int_{\theta}^{\pi/4} \theta\sec^2\theta\,d\theta$$

$$= 2\left[[\theta\tan\theta]_0^{\pi/4} - \int_{0}^{\pi/4} \tan\theta\,d\theta\right]$$

$$= 2\left[\frac{\pi}{4} - \ell n\frac{1}{\sqrt{2}}\sqrt{2}\right] = \frac{\pi}{2} - 2\ell n\sqrt{2} \ = \frac{\pi}{2} - \ell n 2$$

(ii) $$I = \int_{0}^{1} \frac{x\tan^{-1}x}{(1+x^2)^{3/2}}\,dx$$

Put $x = \tan\theta \Rightarrow dx = \sec^2\theta\,d\theta$

$$= \int_{0}^{\pi/4} \frac{\tan\theta\,.\,\sec^2\theta d\theta}{\sec^3\theta} = \int_{0}^{\pi/4} \theta.\sin\theta\,d\theta$$

$$= [-\theta\cos\theta|_0^{\pi/4}] + \sin\theta|_0^{\pi/4}$$

$$= -\frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{4-\pi}{4\sqrt{2}}$$

(iii) $$I = \int_{0}^{1} x^2\sin^{-1}x\,dx$$

Put $\sin^{-1}x = \theta$

$x = \sin\theta$

$dx = \cos\theta\,d\theta$

$$= \int_{0}^{\pi/2} \theta\,.\,\sin^2\theta\cos\theta\,d\theta$$

$$= \theta\,.\left.\frac{\sin^3\theta}{3}\right|_0^{\pi/4} - \frac{1}{3}\int_{0}^{\pi/2} \sin^2\theta\,d\theta$$

$$= \frac{\pi}{2} \cdot \frac{1}{3} - \int_0^{\pi/2} \left(\frac{\sin\theta}{4} - \frac{1}{12}\sin 3\theta \right) d\theta$$

$$= \frac{\pi}{6} - \frac{1}{3} \left[-\frac{\cos\theta}{4} + \frac{\cos 3\theta}{36} \right]_0^{\pi/2} = \frac{\pi}{6} - \frac{2}{9}$$

(iv) $I = \int_0^{\pi/3} \tan^{-1}\left(\frac{2x}{1-x^2} \right) dx$

Put $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$$= \int_0^{\pi/3} \tan^{-1}(\tan 2\theta) \,.\, \sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} 2\theta . \sin^2\theta d\theta + \int_{\pi/4}^{\pi/3} (x - 2\theta)\sec^2\theta\, d\theta$$

$$= \pi \left(1 - \frac{1}{\sqrt{3}} \right) - \ell n 4$$

(iii) $I = \int_0^{\pi/4} \frac{\sin x + \cos x}{4 + 16(\sin 2x)} dx$

$$= \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16[1 - (\sin x - \cos)^2]} dx$$

Put $\sin x - \cos x = t$

$(\cos x + \sin x)\, dx = dt$

$$= \int_{-1}^{0} \frac{dt}{9 + 16(1-t^2)} = \int_{-1}^{0} \frac{dt}{25 - 16t^2} = \frac{1}{16} \int_{-1}^{0} \frac{dt}{\left(\frac{5}{4}\right)^2 - t^2}$$

$$= \frac{1}{16} \cdot \frac{1}{\left(2 \times \frac{5}{4}\right)} \left[\ell n \frac{\frac{5}{4} + t}{\frac{5}{4} - t} \right]_{-1}^{0}$$

$$= \frac{1}{40} \left[-\ell n \frac{\frac{1}{4}}{\frac{9}{4}} \right] = \frac{1}{40} \ell n\, 9 = \frac{1}{20} \ell n 3$$

Sol.6 **(i)** $I = \int_0^{\pi/2} \frac{\sin 2\theta\, d\theta}{\sin^4\theta + \cos^4\theta}$

$$I = \int_0^{\pi/2} \frac{2\sin\theta\cos\theta\, d\theta}{\sin^4\theta + \cos^4\theta} = \int_0^{\pi/2} \frac{2\tan\theta\sec^2\theta}{1 + \tan^4\theta}$$

Put $\tan^2\theta = t$

$2\tan\theta \sec^2\theta\, d\theta = dt$

$$= \int \frac{dt}{1+t^2} = \tan^{-1}(\tan^2\theta)|_0^{\pi/2} = \frac{\pi}{2}$$

(ii) $I = \int_0^{\pi/2} \sqrt{\cos\theta}\, \sin^3\theta\, d\theta$

Put $\cos\theta = t^2$

$\sin\theta\, d\theta = -2t\, dt$

$$= -2\int_1^0 t(\sin^2\theta) t\, dt = -2\int_0^1 t^2(1+t^4) dt$$

$$= -2\left[\frac{t^3}{3} - \frac{t^7}{7} \right]_0^1 = 2\left[\frac{1}{3} - \frac{1}{7} \right] = \frac{7}{1}\ \frac{8}{21}$$

Sol.7 **(i)** $I = \int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}}$

put $x = a\cos^2\theta + b\sin^2\theta$

$dx = (b-a)\sin 2\theta\, d\theta$

Lower limit $a = a\cos^2\theta + b\sin^2\theta$

$\theta = 0$

Upper Limit $b = a\cos^2\theta + b\sin^2\theta$

$b\cos^2\theta + a\cos^2\theta$

$\Rightarrow \cos\theta = 0 \quad \Rightarrow \theta = \frac{\pi}{2}$

$$I = \int_0^{\pi/2} \frac{2(b-a)\sin\theta\cos\theta}{(b-a)\sin\theta\cos\theta} d\theta = \pi$$

(ii) $I = \int_0^b \sqrt{(x-9)(b-x)}\, dx$

Put $x = a\cos^2\theta + b\sin^2\theta$

$dx = (b-a)\sin 2\theta\, d\theta$

$$= 2(b-a)^2 \int_0^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta$$

$$= \frac{(b-a)^2}{2}\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta$$

$$= \frac{(b-a)^2}{2}\int_0^{\pi/2}\sin^2\theta d\theta$$

$$= \frac{(b-a)^2}{2}\int_0^{\pi/2}\left(1-\frac{\cos\theta}{2}\right)d\theta = \frac{x(b-a)^2}{8}$$

Sol.8 **(i)** $I = \int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ By king

$$I = \int_0^{\pi/2}\frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$2I = \int_0^{\pi/2}1.dx \Rightarrow 2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

(ii) $I = \int_0^{\pi/2}\frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}}dx$

Use king

$$I = \int_0^{\pi/2}\frac{e^{\cos x}}{e^{\sin x}+e^{\cos x}}dx$$

Add

$$2I = \int_0^{\pi/2}dx \Rightarrow I = \frac{\pi}{4}$$

(iii) $I = \int_0^{9}\frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx$

King

$$I = \int_0^{a}\frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}\,dx$$

$$2I = \int_0^{a}1.dx \qquad \Rightarrow 2I = a$$

$$I = \frac{a}{2}$$

(iv) $I = \int_0^{\pi/2}\frac{a\sin x+b\cos x}{\sin x+\cos x}dx$

King

$$I = \int_0^{\pi/2}\frac{a\cos x+b\sin x}{\sin x+\cos x}dx$$

Add

$$2I = \int_0^{\pi/2}\frac{a\sin x+b\sin x+a\cos x+b\sin x}{\sin x+\cos x}dx$$

$$= \int_0^{\pi/2}\frac{(a+b)(\sin x+\cos x)}{(\sin x+\cos x)}dx$$

$$2I = (a+b)\,\frac{\pi}{2}$$

$$I = (a+b)\,\frac{\pi}{4}$$

Sol.9 $I = \int_{-1}^{2}\{2x\}dx$

$$2x = t \Rightarrow dx = \frac{dt}{2} = \frac{1}{2}\int_{-2}^{4}\{t\}dt$$

$$= \frac{1}{2}\int_{-2}^{0}\{t\}\,dt + \frac{1}{2}\int_{0}^{4}\{t\}dt$$

$$= \frac{1}{2}\int_{-2}^{0}(t-[t])dt + 2\int_0^1 t\,dt$$

$$= \frac{1}{2}\left.\frac{t^2}{2}\right|_{-2}^{0} - \frac{1}{2}\int_{-2}^{-1}(-2)dt - \int_{-2}^{-1}(-2)dt\ \int_{-1}^{0}(-1)dt+1$$

$$= \frac{1}{4}[0-4] + (-1+2) + \frac{1}{2}\ (OH) + 1$$

$$= -1 + 1 + \frac{1}{2} + 1 = \frac{3}{2}$$

(ii) $I = \int_0^{10\pi} (|\sin x| + |\cos x|)\,dx$

$= \int_0^{10\pi} |\sin x|\,dx + \int_0^{10\pi} |\cos x|\,dx$

$= 10\int_0^{\pi} |\sin x|\,dx + 10\int_0^{\pi} |\cos x|\,dx$

$= 10 \times 2 + 10 \times 2 = 40$

Sol.10 $F(-x) = -f(x)$
$f(x + t) = f(x)$

$\phi(x) = \int_0^{x} f(t)\,dt$

$\phi(x + T) = \int_0^{x+T} f(t)\,dt = \int_0^{x} f(t)\,dt + \int_0^{x+T} f(t)\,dt$

$= \phi(x) + \int_x^{T/2} f(t)\,dt + \int_{T/2}^{2+T} f(t)\,dt$

↑
Sub $u+T=t$
$du=dt$

$= f(x) + \int_x^{T/2} f(t)\,dt + \int_{-T/2}^{x} f(4+T)\,du$

$= \phi(x) + \int_x^{T/2} f(t)\,dt + \int_{-T/2}^{x} f(u)\,du$

$= \phi(x) + \int_x^{T} f(t)\,dt + \int_{-T/2}^{x} f(t)\,dt$

$= \phi(x) + \int_{-T/2}^{T/2} f(t)\,dt \rightarrow$ a as on odd function

$\phi(x + T) = \phi(x)$

Sol.11 $f(x) = 5^{g(x)}$
$f'(x) = 5^{g(g)}\,\ell n5 \,.\, g'(x)$

$g(x) = \int_x^{x^2} \frac{t}{\ln(1+t^2)}\,dt \Rightarrow g'(x) = \frac{x^2}{\ln(1+x^4)}\,.2x$

$f'(\sqrt{2}) = 5^{g(\sqrt{2})}\,\ell n5 \,.\, g'(\sqrt{2}) \Rightarrow g(\sqrt{2}) = 0$

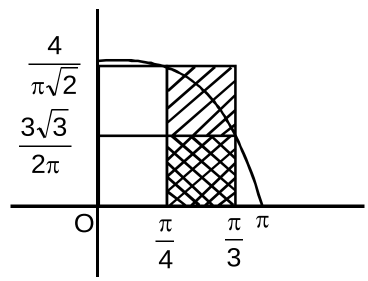
$g'(\sqrt{2}) = \frac{2.2.\sqrt{2}}{\ell n5} = \frac{4\sqrt{2}}{\ell n5}$

$f'(\sqrt{2}) = 1.\ell n5 \,.\, \frac{4\sqrt{2}}{\ell n5} = 4\sqrt{2}$

Sol.12 $f(x) = \int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\,dt$

$f'(x) = \sin^{-1}\sin x\,(2\sin x\cos x)$
$\quad + \cos^{-1}\cos x.\,(-2\cos x\sin x)$
$= x(\sin 2x) + x(-\sin 2x)$
$f'(x) = 0$

Sol.13 (i) P.T. $\frac{\sqrt{3}}{8} < \int_{\pi/4}^{\pi/3} \frac{\sin x}{x}\,dx < \frac{\sqrt{2}}{6}$



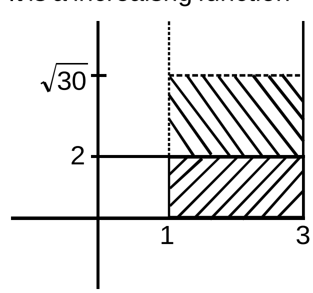
$\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\frac{3\sqrt{3}}{2\pi} < I < \frac{4}{\pi\sqrt{2}}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$\frac{\pi}{12} \times \frac{3\sqrt{3}}{2\pi} < I < \frac{4}{\pi\sqrt{2}} \times \frac{\pi}{12}$

$\frac{\sqrt{3}}{8} < I < \frac{\sqrt{2}}{6}$

(ii) $4 \le \int_1^3 \sqrt{3+x^3}\,dx \le 2\sqrt{30}$

It is a increaisng function



$(3 - 1) \times 2 \le I \le (3 - 1)\sqrt{30}$

$4 \le I \le 2\sqrt{30}$

Sol.14 (i) $\lim\limits_{h\to\infty} \sum\limits_{r=1}^{n-1} \frac{1}{\sqrt{n^2 - r^2}}$

$$\lim_{h\to\infty} \sum_{r=1}^{n-1} \frac{1}{n\sqrt{1-\left(\frac{r}{n}\right)^2}}$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x \,\Big|_0^1$$

$$= \sin^{-1} 1 - 0$$

$$= \frac{\pi}{2}$$

(ii) $\lim\limits_{n\to\infty} \frac{3}{n}\left[1+\sqrt{\frac{n}{n+3}}+\sqrt{\frac{n}{n+6}}+\sqrt{\frac{n}{n+9}}+\ldots+\sqrt{\frac{n}{n+3(n-1)}}\right]$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{r=0}^{n} \sqrt{\frac{n}{n+3r}}$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{r=0}^{n} \sqrt{\frac{2}{1+3\left(\frac{r}{n}\right)}} = 3\int_0^1 \frac{dx}{\sqrt{1+3x}}\, dx = 2$$

Sol.15 $I = \int_0^{\pi} e^{\cos^2 x} \cos^3 (2n+1)x\, dx$

If $n \in$ even integer assume n = 2

$$I = \int_0^{\pi} \underbrace{e^{\cos^2 x} \cos^3 5x\, dx}_{f(x)}$$

$f(\pi - x) = - f(x)$

$I = 0$

If $n \in$ odd integer assume n = 1

$$I = \int_0^{\pi} e^{\cos^2 x} \cos^2 2x\, dx$$

By applying again given $f(\pi - n) = - f(n)$

$I = 0$

Sol.16 $I = \int_0^{a} f(x)g(x)(a-x)h(a-x)dx$

$$= \int_0^{a} f(a-x)g(a-x)h(a-x)\,dx$$

$$I = \int_0^{a} f(x)[-g(x)]\left[\frac{3h(x)-5}{4}\right]dx$$

$$= -\frac{3}{4}\int_0^{a} f(x)g(x)(h(x)\,dx + \frac{5}{4}\int_0^{a} f(x)g(x)\,dx$$

By using again queen

$I = -\frac{3}{4}I \qquad \Rightarrow \quad I = 0$

Sol.17 $\int_0^{x} e^{zx}.e^{-z^2}dz = e^{x^2/4}\int_0^{x} e^{-z^2/4}dz$

$\int_0^{x} e^{(x-z)z}dz \qquad$ $x \to$ constant
$t \to$ Variable

Put $z = \frac{x+t}{2}$

$$= \frac{1}{2}\int_{-x}^{x} e^{\left(\frac{x+t}{2}\right)\left(\frac{x-t}{2}\right)}dt = \frac{1}{2}\int_{-x}^{x} e^{\left(\frac{x^2}{4}-\frac{t^2}{4}\right)}dt$$

$$= \int_0^{x} e^{\frac{x^2}{4}}.e^{-t^2/4}\,dt = e^{x^2/4}\int_0^{x} e^{-t^2/4}dt$$

Sol.18

$$F(x) = \begin{cases} \int_0^x (1-x)\,dx = x - \frac{x^2}{2} \;;\; x \in [0,1] \\ \int_0^1 (1-x)\,dx + \int_1^x 0dx = \frac{1}{2} \;;\; x \in (1,2] \\ \int_0^1 (1-x)\,dx + \int_1^2 0dx + \int_2^x (2-x)^2\,dx = \frac{1}{2} - \frac{1}{3}(2-x)^3 \;;\; x \in (2,3] \end{cases}$$

check the continuity at x = 1, 2

$F(1^-) = \frac{1}{2}$	$F(2^-) = \frac{1}{2}$
$F(1^+) = \frac{1}{2}$	$F(2^+) = \frac{1}{2}$
$F(1) = \frac{1}{2}$	$F(2) = \frac{1}{2}$

Hence F(x) is continuous [0, 3]

check the differentiability at x = 1, 2

$$F'(x) = \begin{cases} 1-x & \text{if } 0 < x < 1 \\ 0 & \text{if } 1 < x < 2 \\ (2-x)^2 & \text{if } 2 < x < 3 \end{cases}$$

$$\begin{array}{l|l} F'(1^-) = 0 & F'(2^-) = 0 \\ F'(1^+) = 0 & F'(2^+) = 0 \\ F'(1) = 0 & F'(2) = 0 \end{array}$$

Hence F(x) is differentiable at $x \in (0, 3)$

Sol.19 There are 3 cases

Case-I when $0 \le x \le 1 \Rightarrow \int_0^1 |x-t| . \cos \pi t \, dt$

$$= \int_0^x (x-t) \cos \pi t \, dt + \int_x^1 (t-x) \cos \pi t \, dt$$

[Using by parts ↑]

$$= -\frac{2\cos \pi x}{\pi^2}$$

Case-II When x < 0 $\Rightarrow \int_0^1 |x-t| . \cos \pi t \, dt$

$$= \int_0^1 (t-x) \cos \pi t \, dt = -\frac{2}{\pi^2}$$

Case-III when x > 1 $\Rightarrow \int_0^1 |x-t| . \cos \pi t \, dt$

$$= \int_1^x (x-t) \cos \pi t \, dt = \frac{2}{\pi^2}$$

Sol.20 $I = \int_0^1 2\sin(pt)\sin(qt)\,dt$

(a) If P & q are diff. roots of the equation tan P = P , tan q = q

$$I = \int_0^1 2\sin pt \, \sin(qt)\,dt$$

Integrates by using by parts taking sin pt as second function.

$$= -\left[2\sin qt \frac{\cos pt}{p}\right]_0^1 + \int_0^1 27\cos t\left(\frac{\cos pt}{p}\right)dt$$

$$= -\frac{2}{p} \sin q \cos p + \frac{2q}{p}\int_0^1 \cos t \cos pt \, dt$$

$$= -\frac{2}{p}\sin q \cos p + \frac{2q}{p}\left[\left[\cos t \frac{\sin pt}{p}\right]_0^1 + \frac{4}{p}\right]_a^1$$

$$+ \frac{4}{p}\int_p^1 \sin p + \sin t \, dt$$

$$= -\frac{2}{p} \sin q \cos p + \frac{2q}{p} \sin po \cos q + \frac{q^2}{p^2} I$$

$$I\left(1 - \frac{q^2}{p^2}\right) = \frac{2q}{p} \sin \cos q - \frac{2}{p} \sin q \cos p$$

(b) Given p = q

$$I = \int_0^1 2\sin^2 p + dt$$

$$= \int_0^1 (1 - \cos 2pt)\,dt = \left[t - \frac{\sin 2pt}{2p}\right]_0^1$$

$$= 1 - \frac{\sin 2p}{2p} = 1 - \frac{(2\tan p)/(1+\tan^2 p)}{2p}$$

$$= 1 - \frac{2p}{(1+p^2)2p} = \frac{p^2}{1+p^2}$$

Sol.21 $f(x) = \frac{\sin x}{x}$

$$f\left(\frac{\pi}{2} - x\right) = \frac{\cos x}{\left(\frac{\pi}{2} - x\right)}$$

$$\frac{\pi}{2}\int_a^{\pi/2} f(x) f\left(\frac{\pi}{2} - \pi\right)dx = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x}{x} \frac{\cos x}{\left(\frac{\pi}{2} - x\right)} dx$$

$$= \pi \int_0^{\pi/2} \frac{\sin 2x}{2x(x-2\pi)} dx$$

$$= \pi \left[\int_0^{\pi/2} \frac{\sin 2x}{2x} dx + \int_0^{\pi/2} \frac{\sin 2x}{x-2x} dx \right]$$

$$= 2 \int_0^{\pi/2} \frac{\sin 2x}{2x} dx$$

Put $2x = t \Rightarrow 2dx = dt$

$$= \int_0^{\pi} \frac{\sin t}{t} dt = \int_0^{\pi} \frac{\sin \pi}{x} dt = \int_0^{\pi} \frac{\sin \pi}{x} dx = \int_0^{\pi} f(x) dx$$

Sol.22. $I = \int_0^1 \frac{1}{(5+2x-2x^2)(1+e^{2-4x})} dx$

Use king's property

$$I = \int_0^1 \frac{dx}{[5+2(1-x)-2(1-x)^2](1+e^{2-4(1-x)})}$$

$$I = \int_0^1 \frac{dx}{(5+2x-2x^2)(1+e^{-(2-4x)})} dx$$

$$2I = \int_0^1 \frac{e^{2-4x}+1}{(5+2x-2x^2)(1+e^{2-1x})} dx$$

$$2I = \int_0^1 \frac{dx}{5+2x-2x^2} = \frac{1}{4} \int_0^1 \frac{dx}{\frac{5}{2}+x-x^2}$$

$$= \frac{1}{4} \int_0^1 \frac{dx}{\frac{11}{4} - \left(x - \frac{1}{2}\right)^2}$$

$$= \frac{1}{\sqrt{11}} \ell n \left(\frac{\sqrt{11}+1}{\sqrt{11}-1} \right)$$

Sol.23 $I = \int_0^{\infty} \frac{dx}{(x + \_\sqrt{1+x^2})^n}$

Put $x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

$$= \int_0^{\pi/2} \frac{\sec^2\theta \, d\theta}{(\tan\theta + \sec\theta)^n} = \int_0^{\pi/2} \frac{\cos^{n-2}\theta \, d\theta}{(1+\sin\theta)^n}$$

King

$$I = \int_0^{\pi/2} \frac{\sin^{n-2}\theta \, d\theta}{(1+\cos\theta)^n} = \int_0^{\pi/2} \frac{\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{n-2} d\theta}{\left(2\cos^2\frac{\theta}{2}\right)^n}$$

$$= \frac{1}{4} \int_0^{\pi/2} \frac{\sin^{n-2}(\theta/2)}{\cos^{n+2}(\theta/2)} d\theta$$

$$= \frac{1}{4} \int_0^{\pi/2} \frac{\sin^{n-2}(\theta/2)}{\cos^{n-2}(\theta/2)} \cdot \frac{1}{\cos^4(\theta/2)} d\theta$$

$$I = \frac{1}{4} \int_0^{\pi/2} \tan^{n-2}\left(\frac{\theta}{2}\right)\left(1+\tan^2\frac{\theta}{2}\right)\sec^2\left(\frac{\theta}{2}\right) d\theta$$

put $\tan\frac{\theta}{2} = t$

$\frac{1}{2}\sec^2\frac{\theta}{2} \, d\theta = dt$

$$= \frac{1}{2} \int_0^1 t^{n-2}(1+t^2) dt = \frac{1}{2} \int_0^1 (t^{n-2}+t^n) dt$$

$$= \frac{1}{2}\left[\frac{t^{n-1}}{n-1} + \frac{t^{n+1}}{n+1}\right]_0^1 = \frac{1}{2}\left[\frac{1}{n-1} + \frac{1}{n+1}\right] = \frac{n}{n^2-1}$$

Sol.24 $\int_0^1 (\{2x\}-1)(\{3x\}-1) \, dx,$

$$= \int_0^1 (2x-[2x]-1)(3x-[3x]-1) \, dx$$

$$= \int_0^{1/3} (2x-1)(3x-1)\,dx + \int_{1/3}^{1/2} (2x-1)(3x-2)\,dx$$

$$+ \int_{1/2}^{2/3} (2x-2)(3x-2)\,dx + \int_{2/3}^{1} (2x-2)(3x-3)\,dx$$

$$= \frac{19}{72}$$

Sol.26 $g(x) = \int_x^a \frac{f(t)}{t}\,dt$

$$g'(x) = -\frac{f(x)}{x}$$

$x\,g'(x) = -f'(x)$

Integrate both the side w.r.t.x.

$$\int_0^a x.g'(x)\,dx = -\int_0^a f(x)\,dx$$

$$x\,g(x)\big|_0^a - \int_0^a g(x)\,dx = -\int_0^a f(x)\,dx$$

$$g(x) - \int_0^a g(x)\,dx = -\int_0^a f(x)\,dx$$

$$g(x) = \int_0^a \frac{f(t)}{t}\,dt = 0$$

$$g(a) = \int_0^a \frac{f(t)}{t}\,dt = 0$$

$$-\int_0^a g(x)\,dx = -\int_0^a f(x)\,dx$$

$$\int_0^a g(x)\,dx = \int_0^a f(x)\,dx$$

Sol.27 $I = \int_0^x \frac{x\,dx}{9\cos^2 x + \sin^2 x}$

$$I = \int_0^x \frac{(\pi - x)dx}{9\cos^2 x + \sin^2 x} \text{ king}$$

add

$$2I = \pi \int_0^{\pi} \frac{dx}{9\cos^2 x + \sin^2 x}$$

queen

$$2I = 2\pi \int_0^{\pi/2} \frac{dx}{9\cos^2 x + \sin^2 x}$$

$$I = \pi \int_0^{\pi/2} \frac{dx}{9\cos^2 x + \sin^2 x}$$

$$I = \pi \int_0^{\pi/2} \frac{\sec^2 x\,dx}{\tan^2 x + 9}$$

Put $\tan x = t \quad \Rightarrow \quad \sec^2 x\,dx = dt$

$$= \pi \int_0^{\infty} \frac{dt}{a + t^2} = \frac{\pi}{3} \tan^{-1} \frac{t}{3}\Big|_0^{\infty} = \frac{\pi}{3}\left[\frac{x}{2}\right] = \frac{x^2}{6}$$

Sol.28 $I = \int_0^{x/2} \sqrt{\frac{1-\sin 2x}{1+\sin 2x}}\,dx$

$$= \int_0^{x/2} \sqrt{\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)}\,dx = \int_0^{x/2} \left|\frac{\cos x - \sin x}{\cos x + \sin x}\right| dx$$

By using queen properly

$$I = 2\int_0^{x/2} \left(\frac{\cos x - \sin x}{\cos x + \sin x}\right) dx$$

Let $\cos x + \sin x = t$

$(\cos x - \sin x)\,dx = dt$

$$= 2\int_1^{\sqrt{2}} \frac{dt}{t} = 2\,[\ell nt]_1^{\sqrt{2}}$$

$I = 2\,[\ell n \sqrt{2}\,] \quad \Rightarrow \quad I = \ell n2$

Sol.29 $I_n = \int_1^e (\ell nx)^n\,dx$

$= x\,(\ell nx)\Big|_1^e - \int_1^e n(\ell nx)^{n-1}.\frac{1}{x}.x\,dx$

$I_n = e(\ell ne)^n - n\,I_{n-1}$

$I_n + n\,I_{n-1} = e$

Put n = 1, 2, 3 respectively

$I_3 + 3I_2 = e$ ...(1)

$I_2 + 2I_1 = e$ ...(2)

$I_1 + I_0 = e$ ...(3)

$I_0 = e - 1$

$I_1 = 1$

$I_2 = 0 - 2$

$I_3 = 6 - 2e$

Sol.30 $I = \int_0^{\pi/2} 2\sin x\cos x\tan^{-1}(\sec x)\,dx$

Put sinx = t

cosx dx = dt

$= 2\int_0^1 t(\tan^{-1}x)\,dt \;=\; 2\left[\frac{t^2}{2}\tan^{-1}t\int\frac{t^2}{1+t^2}\,dt\right]$

$= t^2\tan^{-1}t - \int 1.dt\int\frac{dt}{1+t^2}$

$= t^2\tan^{-1}t - t + \tan^{-1}t\,\Big|_0^1 = \frac{\pi}{2} - 1$

Sol.31 $I = \int_0^{\pi/4} \frac{x\,dx}{\cos x(\cos x + \sin x)}$

$= \int_0^{\pi/4}\frac{x\,dx}{\cos^2 x + \cos x\sin x} = \int_0^{\pi/4}\frac{x\,dx}{\frac{1+\cos 2x}{2}+\frac{\sin 2x}{2}}$

$I = 2\int_0^{\pi/4}\frac{x\,dx}{1+\cos 2x+\sin 2x}$

King's

$I = 2\int_0^{\pi/4}\frac{\left(\frac{3}{4}-x\right)dx}{1+\sin 2x+\cos 2x}$

$2I = 2 \times \frac{\pi}{4}\int_0^{\pi/4}\frac{dx}{1+\sin 2x+\cos 2x}$

$I = \frac{\pi}{4}\int_0^{\pi/4}\frac{dx}{2\cos^2 x + 2\sin x\cos x}$

$= \frac{\pi}{4}\int_0^{\pi/4}\frac{dx}{2\cos^2 x(1+\tan x)}dx$

put 1 + tanx = t

$\sec^2 x\,dx = dt$

$= \frac{\pi}{8}\,\ell n\,(1+\tan x)\Big|_0^{\pi/4}$

$I = \frac{\pi}{8}\,\ell n2$

Sol.32 $I = \int_1^2 \frac{(x^2-1)\,dx}{x^3\sqrt{(x^2)^2+(x^2-1)^2}}$

$= \int_1^2 \frac{dx}{x^3\sqrt{\left(\frac{x^2}{x^2-1}\right)^2+1}}$

Let $\frac{x^2}{x^2-1} = t$

$\frac{x^2-1}{x^2} = \frac{1}{t}$

$1 - \frac{1}{x^2} = \frac{1}{t}$

$\frac{2}{x^3}\,dx = -\frac{1}{t^2}\,dt$

$\pm - \int_\infty^{4/3}\frac{dt}{2t^2\sqrt{t^2+1}} = \int_\infty^{4/3}\frac{dt}{2t^3\sqrt{1+\frac{1}{t^2}}}$

Let $1 + \frac{1}{t^2} = u \Rightarrow -\frac{2}{t^3}\,dt = du$

$$= \frac{1}{4}\int_1^{25/16}\frac{du}{\sqrt{u}} = \frac{1}{4}\,[2\sqrt{u}]_1^{25/16}$$

$$= \frac{1}{2}\left[\sqrt{\frac{25}{16}}-1\right] = \frac{1}{2}\left(\frac{5}{4}-1\right) = \frac{1}{8} \Rightarrow \frac{u}{v} = \frac{1}{8}$$

$$(1000) \times \frac{u}{v} = \frac{1000}{6} = 125$$

Sol.33 $I = \int_0^{\pi}|\sqrt{2}\sin x + 2\cos x|\,dx$

$\sqrt{2}$ sinx + 2 cosx = 0

tanx = – $\sqrt{2}$

$$= \int_0^{\pi-\tan^{-1}\sqrt{2}}(\sqrt{2}\sin x + 2\cos x)\,dx - \int_{\pi-\tan^{-1}\sqrt{2}}^{\pi}(\sqrt{2}\sin x + 2\cos x)\,dx$$

$$= [-\sqrt{2}\cos x + 2\sin x]_0^{\pi-\tan^{-1}\sqrt{2}} - [-\sqrt{2}\cos x + 2\sin x]_{\pi-\tan^{-1}\sqrt{2}}^{0}$$

$= \sqrt{2}\cos(\tan^{-1}\sqrt{2}) + \sin(\tan^{-1}\sqrt{2}) - (-\sqrt{2}+0)$

$+ \sqrt{2}\cos x + 2\sin x + \sqrt{2}\cos(\tan^{-1}\sqrt{2})$

$+ 2\sin(\tan^{-1}\sqrt{2})$

$$= \sqrt{2}\,\frac{\sqrt{1}}{\sqrt{3}} + 2\sqrt{\frac{2}{3}} + \sqrt{2} - \sqrt{2} + 0 + \frac{\sqrt{2}}{\sqrt{3}} + 2\frac{\sqrt{2}}{\sqrt{3}}$$

$$= 6\sqrt{\frac{2}{3}} = 2\sqrt{6}$$

Sol.34 $I = \int_3^5\Big(\underbrace{\sqrt{x+2\sqrt{2x-4}}+\sqrt{x-2\sqrt{2x-4}}}_{f(x)}\Big)dx$

$f^2(x) = x + 2\sqrt{2x-4} + x - 2\sqrt{2x-4} + 2\sqrt{x^2-4(2x-4)}$

$f^2(x) = 2x + 2\sqrt{x^2-8x+16}$

$f^2(x)$ $2x + 2(x-4) = 4x - 8$

$f(x) = 2\sqrt{x-2}$

$$I = \int_3^5 2\sqrt{x-2}\,dx = 2\left.\frac{(x-2)^{3/2}}{\frac{3}{2}}\right|_3^5$$

$$= \frac{4}{3}[(3)^{3/2} - 1] = \frac{4}{3}[3\sqrt{3} - 1] = 4\sqrt{3} - \frac{4}{3}$$

Sol.35 $P = \int_0^{\infty}\frac{x^2}{1+x^4}dx$

$R = \int_0^{\infty}\frac{dx}{1+x^4}$ ...(1)

pout $x = \frac{1}{t}$ $\Rightarrow dx = -\frac{dt}{t^2}$

$$= -\int_{\infty}^{0}\frac{t^2}{1+t^4}dt = \int_0^{\infty}\frac{t^2}{1+t^4}dt = P$$

P = R

$I = \int_0^{\infty}\frac{x^2dx}{1+x^4}$ ...(2)

add (1) + (2)

$$2I = \int_0^{\infty}\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx = \int_0^{\infty}\frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)+2}dx$$

Put $x - \frac{1}{x} = t \Rightarrow \left(1+\frac{1}{x}\right)dx = dt$

$$= \int\frac{dt^2}{t^2(\sqrt{2})^2} = \frac{1}{\sqrt{2}}\tan^{-1}\left.\frac{x-\frac{1}{x}}{\sqrt{2}}\right|_0^{\infty} = \frac{1}{\sqrt{2}}\left[\frac{\pi}{2}\right]$$

$a = \int_0^{\infty}\frac{x\,dx}{1+x^4}$ Put $x^2 = t \Rightarrow x\,dx = \frac{dt}{2}$

$$= \frac{1}{2}\int_0^{\infty}\frac{dt}{1+t^2} = \frac{1}{2}[\tan^{-1}t]_0^{\infty} = \frac{\pi}{4}$$

$$P - \sqrt{2}\ 2 + R = \frac{\pi}{2\sqrt{2}}$$

Sol.36 $= \int_0^1\left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{x^2+1}\right)dx$

$$I = \left[\frac{x^7}{7} - \frac{4x^6}{6} + x^5 - \frac{4x^3}{3} + 4x - 4\tan^{-1}x\right]_0^1$$

$$I = \frac{22}{7} - \pi$$

Sol.37 $I = \int_0^1 \frac{x^2 . \ln x}{\sqrt{1-x^2}}\, dx$ put x = sin θ, dx = cos dθ

$I = \int_0^{\pi/2} \sin^2\theta\, \ell n \sin\theta\, d\theta$

$I = \int_0^{\pi/2} \left(\frac{1-\cos 2\theta}{2}\right) \ell n \sin\theta\, d\theta$

$= \frac{1}{2}\int_0^{\pi/2} \ell n \sin\theta\, d\theta - \frac{1}{2}\int_0^{\pi/2} \cos 2\theta\, \ell n \sin\theta\, d\theta$

$= \frac{\pi}{8}\ (1 - \ell n\, 4)$

Sol.38 $I = \int_{-2}^{2} \frac{x^2\, dx}{\sqrt{x^2+4}} - \int_{-2}^{2} \frac{x\, dx}{\sqrt{x^2+4}} \rightarrow 0$ as it is an odd function

$= 2\int_3^2 \frac{x^2 dx}{\sqrt{x^2+4}} = 2\int_0^2 \frac{x^2+4-4}{\sqrt{x^2+4}}\, dx$

$= 2\int_0^2 \sqrt{x^2+4}\, dx - 8\int_0^2 \frac{dx}{\sqrt{x^3+4}}$

$= 2\left[\frac{x}{2}\sqrt{x^2+4} + 2\ell n(x+\sqrt{x^2+4})\right]_0^2 - 8\,\ell n(x+\sqrt{x^2+4})\,\Big|_0^2$

$= 4\sqrt{2} - 4\ell n\,(\sqrt{2} + 1)$

Sol.39 $I = \int_0^1 2\tan^{-1} x\, dx + \int_1^{\sqrt{3}} (\pi + 2\tan^{-1} x)\, dx$

using by parts

$= \frac{\pi\sqrt{3}}{3}$

Sol.40 $I = \int_0^{\pi/2} \frac{a \sin x + b \cos x}{\sin\left(\frac{x}{4}+\pi\right)} dx$

$I = \sqrt{2} \int_0^{\pi/2} \frac{a \sin x + b \cos x}{\sin x + \cos x} dx$

by king's property

$I = \sqrt{2} \int_0^{\pi/2} \frac{a \cos x + b \sin x}{\sin x + \cos \pi} dx$

$2I = \sqrt{2} \int_0^{\pi/2} \frac{a \cos x + b \sin x}{\sin x + \cos x}\, dx$

$I = \frac{(a+b)\pi}{4\sqrt{2}}$

Sol.41 $I = \frac{1}{2}\int_0^{4\pi} \frac{dx}{2+\sin x}$ ....(1)

$I = \frac{1}{2}\int_0^{4\pi} \frac{dx}{2-\sin x}$ ....(2)

$2I = \frac{1}{2}\int_0^{4\pi} \frac{4}{4-\sin^2 x}\, dx$

$I = 4\int_0^{\pi} \frac{dx}{4-\sin^2 x} \Rightarrow I = 8\int_0^{\pi/2} \frac{dx}{4-\sin^2 x}$

$I = 8\int_0^{\pi/2} \frac{\sec^2 x\, dx}{4+3\tan^2 x} \Rightarrow I = 8\int_0^{\infty} \frac{dt}{3t^2+4} = \frac{2\pi}{\sqrt{3}}$